Baseline Geometry and EOP

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October 28, 1998

Abstract

This note derives the normal equations and formal errors for single baseline EOP. Since any multi-baseline experiment can be built up from a set of single-baseline experiments these results are quite general. We examine with particular interest the case where all of the stations lie on a great circle, so that the baselines are coplanar. VLBI lore has it that this is a bad network. We show that this is not the case.

1 Introduction

In this note I derive analytic formulas for the EOP formal errors as a function of baseline geometry. In order to do this, I make the following simplifying assumptions. 1) The sky is uniformly sampled; 2) The number of observations is large; 3) The observation errors are independent of the observation. This allows me to replace various sums by integrals which can be done exactly.

I start by looking at a single-baseline experiment on the equator I derive the normal equations, and the formal errors for the EOP components. I then show how this result can be written for an arbitrary baseline in terms of the "LTV" baseline unit vectors. Since any VLBI measurement can be considered to be a sum of single baseline experiments, this allows me to determine the formal errors for an arbitrary VLBI network in principle.

The last thing I look at is the EOP determined by a regular polygon. There is an old saw in VLBI EOP circles is that you can't determine EOP very well if all stations lie on a circle. I show this is wrong by looking at the case of 3 stations on the vertices of an equalateral triangle, and N-equally spaced stations on a great circle.

2 Single Baseline EOP

In this section I derive the formal errors for single baseline EOP. For simplicity, I assume that the two stations are on the equator, and are placed symmetrically at longitudes $\pm\frac{\alpha}{2}$ so that their coordinates are given by:

$$\vec{R}_{\pm} = R(\cos\frac{\alpha}{2}, \pm\sin\frac{\alpha}{2}, 0)$$

I can derive the results for other stations at other locations by a simple transformation of the normal equations. The baseline vector is then:

$$\begin{aligned} \vec{R}_{bl} &= 2R(0, \sin\frac{\alpha}{2}, 0) \\ &= 2R\sin\frac{\alpha}{2}\hat{y} \end{aligned}$$

hence $R_{bl} = 2R\sin\alpha$. The unit vector in the direction of some source is

$$\widehat{S} = (\cos\Theta\cos\varphi, \cos\Theta\sin\varphi, \sin\Theta)$$

where Θ is the latitude, and φ the longitude. From the site vector and the source vector it is easy to construct the local elevation and azimuth angles. The sine of the elevation is given by:

$$\begin{aligned} \sin EL_{\pm} &= \widehat{R}_{\pm} \cdot \widehat{S} \\ &= \cos\Theta\cos(\varphi \mp \frac{\alpha}{2}) \end{aligned}$$

The condition that a source be visible is that $\sin EL \geq 0$. Actually, we don't usually observe down to the horizon. Instead we observe down to some minimum elevation. Hence the observations that we use satisfy the condition:

$$\sin EL \geq \sin EL_{\min}$$

In order to use a given source it must be above the elevation limit at both sites. For simplicity we assume that both elevation limits are the same. Then we can observe a source at both antennas if both of the following conditions hold:

$$\begin{aligned} \cos\Theta\cos(\varphi - \frac{\alpha}{2}) &\geq \sin EL_{\min} \\ \cos\Theta\cos(\varphi + \frac{\alpha}{2}) &\geq \sin EL_{\min} \end{aligned}$$

The VLBI delay is modeled as:

$$\tau = \widehat{S} \cdot \vec{R}_{bl}$$

A rotation can be specified by a three vector $\overrightarrow{\omega}$ where the first component gives the delay about the x-axis, etc. Under an infinitesimal rotation the delay transforms according to:

$$\tau = \widehat{S} \cdot (\vec{R}_{bl} + \overrightarrow{\omega} \times \vec{R}_{bl})$$

The partial derivative of the delay with respect to $\overrightarrow{\omega}$ is given by:

$$\frac{\partial\tau}{\partial\overrightarrow{\omega}} = \widehat{S} \times \vec{R}_{bl}$$

This result is true for an arbitrary source and baseline. For our particular choice of baseline we have:

$$\frac{\partial\tau}{\partial\overrightarrow{\omega}} = R_{bl}(-\sin\Theta, 0, \cos\Theta\cos\varphi)$$

Note that the partial derivative of the y-component is 0. This makes sense, because rotations about the axis of the baseline, in this case, the y-direction, leave the delay invariant. Hence we are not sensitive to this component of EOP. There are 4 potentially non-vanishing components of the normal matrix N:

$$\begin{aligned} N_{xx} &= R_{bl}^2 \sum_{obs} (\sin\Theta_j)^2 \frac{1}{\sigma_j^2} \\ N_{zz} &= R_{bl}^2 \sum_{obs} \left(\cos\varphi_j \cos\Theta_j\right)^2 \frac{1}{\sigma_j^2} \\ N_{xz} &= N_{zx} = -R_{bl}^2 \sum_{obs} \left(\sin\Theta_j \cos\varphi_j \cos\Theta_j\right) \frac{1}{\sigma_j^2} \end{aligned}$$

Here N_{xx} etc. is the normal equation matrix element for the Xpole-Xpole and the sum is over all of the observations. If we assume that 1.) we have lots of observations 2) that they are uniformly distributed and 3) the observation errors are constant, then we can replace the above sums by integrals:

$$\sum_{obs} f(\Theta_j, \varphi_j) \Rightarrow \frac{1}{\sigma^2} \frac{Num\_obs}{\int d\Omega} \int f(\Theta, \varphi) d\Omega$$

Here the integration is over all parts of the sky which are mutually visible. The factor in front is a normalization factor which insures we get the correct value (Num_obs/σ^2) if $f(\Theta, \varphi) = 1$. This normalization factor is the same for all matrix elements. Since all I am interested in is the relative size of formal errors, this can be ignored. With these observations, the sums can be replaced by integrals:

$$\begin{aligned} N_{xx} &= R_{bl}^2 \int d\Omega \, (\sin\Theta)^2 \cos\Theta \\ N_{zz} &= R_{bl}^2 \int d\Omega \, (\cos\varphi \cos\Theta)^2 \cos\Theta \\ N_{xz} &= N_{zx} = -R_{bl}^2 \int d\Omega \, (\sin\Theta \cos\varphi \cos\Theta) \cos\Theta \end{aligned}$$

where the angular integration $\int d\Omega$ is restricted to that part of the sky which is mutually visible to both sites. The third integral vanishes since the contribution of $\sin\Theta$ on opposite sides of the equator cancels out. We now turn to valuating the other two integrals.

If the cutoff angle is 0, then the integrations can be done explicitly. The mutually visibility conditions become:

$$\begin{aligned} \cos(\varphi - \frac{\alpha}{2}) &\geq 0 \\ \cos(\varphi + \frac{\alpha}{2}) &\geq 0 \end{aligned}$$

which can be re-written as:

$$\begin{aligned} -\pi/2 &\leq \varphi - \frac{\alpha}{2} \leq \pi/2 \\ -\pi/2 &\leq \varphi + \frac{\alpha}{2} \leq \pi/2 \end{aligned}$$

which has the solution

$$\frac{\alpha - \pi}{2} \leq \varphi \leq \frac{\pi - \alpha}{2}$$

The first integral then becomes:

$$\begin{aligned} N_{xx} &= R_{bl}^2 \int_{\frac{\alpha-\pi}{2}}^{\frac{\pi-\alpha}{2}} d\varphi \int_{-\pi/2}^{\pi/2} (\sin\Theta)^2 \cos\Theta d\Theta \\ &= \frac{2}{3} R_{bl}^2 \, (\pi - \alpha) \end{aligned}$$

from which it follows that

$$\begin{aligned}
\sigma_x &= \sqrt{\frac{1}{N_{xx}}} \\
&= \frac{1}{R_{bl}}\sqrt{\frac{3}{2}}\sqrt{\frac{1}{\pi-\alpha}}
\end{aligned}$$

The second integral becomes:

$$\begin{aligned}
N_{zz} &= R_{bl}^2 \int\limits_{-(\pi/2-\alpha)}^{\pi/2-\alpha} \cos^2\varphi d\varphi \int\limits_{-\pi/2}^{\pi/2} (\cos\Theta)^2 \cos\Theta d\Theta \\
&= \frac{2}{3} R_{bl}^2 \left(\pi - \alpha + \sin\alpha\right)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\sigma_z &= \sqrt{\frac{1}{N_{33}}} \\
&= \frac{1}{R_{bl}}\sqrt{\frac{3}{2}}\sqrt{\frac{1}{\pi-\alpha+\sin\alpha}}
\end{aligned}$$

Note that the formal errors for the z component of EOP are always smaller than for the x component. The ratio of the two is:

$$\frac{\sigma_z}{\sigma_x} = \sqrt{\frac{1}{1+\frac{\sin\alpha}{\pi-\alpha}}}$$

If we are interested in the “normalized” normal equations, we need to calculate the normalization factor, which is easy to do:

$$\begin{aligned}
\int d\Omega &= \int\limits_{\frac{\alpha-\pi}{2}}^{\frac{\pi-\alpha}{2}} d\varphi \int\limits_{-\pi/2}^{\pi/2} \cos\Theta d\Theta \\
&= 2(\pi-\alpha)
\end{aligned}$$

Hence the normalized normal matrix elements are:

$$\begin{aligned}
N_{xx}(normalized) &= \frac{1}{3}\frac{Num_obs}{\sigma^2} R_{bl}^2 \\
N_{zz}(normalized) &= \frac{1}{3}\frac{Num_obs}{\sigma^2} R_{bl}^2 \left(1 + \frac{\sin\alpha}{\pi-\alpha}\right)
\end{aligned}$$

and the formal errors go like:

$$\begin{aligned}
\sigma_x &= \sqrt{\frac{3}{Num_obs}}\frac{\sigma}{R_{bl}} \\
\sigma_z &= \sqrt{\frac{3}{Num_obs}}\frac{\sigma}{R_{bl}}\sqrt{\frac{1}{1+\frac{\sin\alpha}{\pi-\alpha}}}
\end{aligned}$$

The ratio of the formal errors remains the same. The separation between the stations varies between 0 and 180 degrees. The ratio of the formal errors varies between 1 and $1/\sqrt{2}$ as illustrated in the figure below:

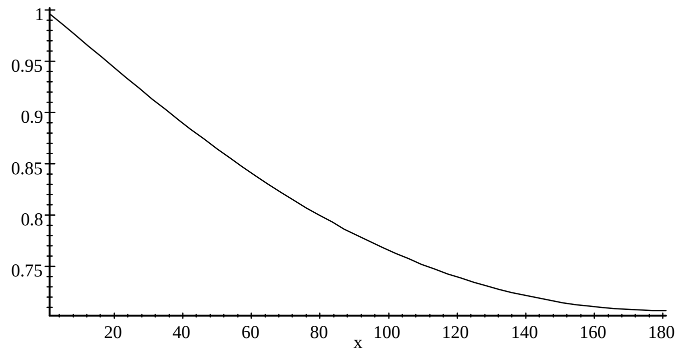

Figure 1. Ratio of formal errors for EOP vs separation in degrees

3 Coordinate Independent Form of Normal Matrix

In the previous section I derived the EOP normal equation for a single baseline. In this section I recast it in a form which is suitable for use for arbitrary baselines. Consider an arbitrary baseline defined by two stations $\overrightarrow{r_i}$ and $\overrightarrow{r_j}$. This baseline has a natural orthogonal triplet of vectors associated with it:

$$
\begin{aligned}
\widehat{e}_{L,ij} &= \frac{\overrightarrow{r_i} - \overrightarrow{r_j}}{|\overrightarrow{r_i} - \overrightarrow{r_j}|} \\
\widehat{e}_{V,ij} &= \frac{\overrightarrow{r_i} + \overrightarrow{r_j}}{|\overrightarrow{r_i} + \overrightarrow{r_j}|} \\
\widehat{e}_{T,ij} &= \frac{\overrightarrow{r_i} \times \overrightarrow{r_j}}{|\overrightarrow{r_i} + \overrightarrow{r_j}|\,|\overrightarrow{r_i} - \overrightarrow{r_j}|}
\end{aligned}
$$

These are the "LVT" unit vectors–length, vertical and transervese unit vectors. Assuming uniform sky coverage, the contribution of this baseline to the EOP normal matrix is given by:

$$N = N_{VV}\widehat{e}_{V,ij} \otimes \widehat{e}_{V,ij} + N_{TT}\widehat{e}_{T,ij} \otimes \widehat{e}_{T,ij}$$

where

$$
\begin{aligned}
N_{VV} &= \frac{1}{3}\frac{Num\_obs}{\sigma^2}|\overrightarrow{r_i} - \overrightarrow{r_j}|^2 \\
N_{TT} &= \frac{1}{3}\frac{Num\_obs}{\sigma^2}|\overrightarrow{r_i} - \overrightarrow{r_j}|^2\left(1 + \frac{\sin\alpha}{\pi - \alpha}\right)
\end{aligned}
$$

The angle α is the angle between the two stations. It is obvious from this that the tranverse component of EOP is better determined than the vertical component. This formula allows us to calculate the EOP normal equations for an arbitrary baseline. If we have several baselines we just sum over all of them.

4 Baselines Spaced Along an Equilateral Triangle

We now turn to the special case of three stations equally spaced along the equator. For simplicity, we assume that they are spaced at ± 60, and 180. Call the three stations $\overrightarrow{R}_{\pm}$ and $\overrightarrow{R}_{180}$. We have three stations are at:

$$\begin{aligned}
\overrightarrow{R}_{\pm} &= r(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 0) \\
\overrightarrow{R}_{180} &= r(-1, 0, 0)
\end{aligned}$$

The associated baselines are:

$$\begin{aligned}
\overrightarrow{R}_{+-} &= \overrightarrow{R}_{+} - \overrightarrow{R}_{-} = r\sqrt{3}(0, 1, 0) \\
\overrightarrow{R}_{180\pm} &= \overrightarrow{R}_{180} - \overrightarrow{R}_{+-} = r(\frac{3}{2}, \mp\frac{\sqrt{3}}{2}, 0)
\end{aligned}$$

The baseline vertical unit vectors for the three baselines are:

$$\begin{aligned}
\widehat{e}_{V,+-} &= (1, 0, 0) \\
\widehat{e}_{V,180\pm} &= (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 0)
\end{aligned}$$

The baseline transverse unit vector is the same for all three baselines, and is given by:

$$\widehat{e}_T = (0, 0, 1)$$

The total EOP normal equation is given by the sum:

$$N_{tot} = N_{VV}\left(\widehat{e}_{V,+-} \otimes \widehat{e}_{V,+-} + \widehat{e}_{V,180+} \otimes \widehat{e}_{V,180+} + \widehat{e}_{V,180-} \otimes \widehat{e}_{V,180-}\right) + 3N_{TT}\widehat{e_T \otimes e_T}$$

Which, after simplification, takes the form:

$$N_{tot} = \begin{pmatrix} \frac{3}{2}N_{VV} & 0 & 0 \\ 0 & \frac{3}{2}N_{VV} & 0 \\ 0 & 0 & 3N_{TT} \end{pmatrix}$$

The formal errors for both Xpole and Ypole are the same, and are given by

$$\begin{aligned}
\sigma_x &= \sigma_y = \frac{1}{R_{bl}}\sqrt{\frac{1}{\pi - \alpha}} \\
&= \frac{1}{R_{bl}}\sqrt{\frac{3}{\pi}}
\end{aligned}$$

where I have used $\alpha = \pi/6$. The formal errors for the Z component (UT1) are:

$$\begin{aligned}
\sigma_z &= \frac{1}{R_{bl}}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{\pi - \alpha + \sin\alpha}} \\
&= \frac{1}{R_{bl}}\sqrt{\frac{3}{2\pi + 3\sqrt{3}}}
\end{aligned}$$

The ratio of the formal errors is:

$$\begin{aligned}\frac{\sigma_z}{\sigma_x} &= \sqrt{\frac{\pi}{2\pi + 3\sqrt{3}}} \\ &= 0.52314\end{aligned}$$

Hence for this case UT1 is determined roughly twice as well as X or Y pole.

5 EOP accuracy for a regular polyhedron.

For the more general case where we have n stations spaced around the equator and you assume that stations observed only with their nearest neighbor, you can show that the normal matrix takes the form:

$$N_{tot} = \begin{pmatrix} \frac{n}{2}N_{VV} & 0 & 0 \\ 0 & \frac{n}{2}N_{VV} & 0 \\ 0 & 0 & nN_{TT} \end{pmatrix}$$

The X and Y pole formal errors are equal, and go like

$$\begin{aligned}\sigma_x &= \sigma_y = \sqrt{\frac{2}{n}}\frac{1}{R_{bl}}\sqrt{\frac{3}{2}}\sqrt{\frac{1}{\pi - \alpha}} \\ &= \frac{1}{R_{bl}}\sqrt{\frac{3}{n-2}\frac{1}{\pi}}\end{aligned}$$

where I have used $\alpha = 2\pi/n$. The formal errors for the UT1 are:

$$\begin{aligned}\sigma_z &= \sqrt{\frac{1}{n}}\frac{1}{R_{bl}}\sqrt{\frac{3}{2}\frac{1}{\pi - \alpha + \sin\alpha}} \\ &= \frac{1}{R_{bl}}\frac{1}{\sqrt{2}}\sqrt{\frac{3}{\pi(n-2) + n\sin\frac{2\pi}{n}}}\end{aligned}$$

while the ratios of the errors are:

$$\frac{\sigma_z}{\sigma_x} = \frac{1}{\sqrt{2}}\sqrt{\frac{\pi(n-2)}{\pi(n-2) + n\sin\frac{2\pi}{n}}}$$

hence the formal errors for the UT1 component are smaller by at least a factor $\sqrt{2}$ then for the polar motion components. This conclusion is true even if we relax the assumption of nearest neighbor observation. Figure 2 below plots the ratio of the formal errors as a function of number of vertices of the polyhedron. This ratio is generally close to 0.5.

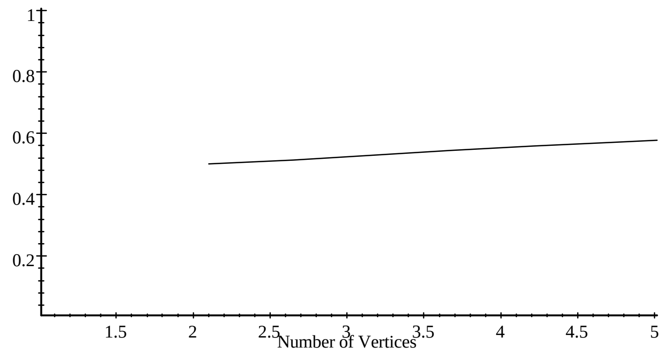

Figure 2. Ratio of EOP formal errors for regular polyhedron.